W.E. Baylis – Electrodynamics - Chapter 1 – Solutions to selected Problems

P 1.1

Galilean transformations relate frames moving at constant velocity with respect to one another when it is assumed there exists a universal time valid for all frames. Under Galilean transformations between frame *A* and another frame *B* in which *A* is moving with constant velocity V, a velocity $\mathbf{v}_A$ seen in frame A is seen as $\mathbf{v}_B = \mathbf{v}_A + \mathbf{V}$ in frame *B*.

a) Show, assuming validity of Newton's second law and the Galilean transformation above of velocities, that forces are the same in frames *A* and *B*.

b) Use the result of part a) together with the Lorentz-force law $\mathbf{f} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}$ to derive the Galilean transformation law force for electric and magnetic fields.

c) Consider a very long line of charge at rest in frame *A* and use Gauss's law to find the fields. Let λ be the charge per unit length along the line. Now consider the source as seen in frame B when the relative velocity **V** lies along the line of charge. Find the fields as seen in frame *B*. Are they consistent with the transformation law found in b) ?

Sol P1.1

a)

mass invariant

$$\overset{\in\mathbb{R}^3}{F_A} = \frac{d}{dt}(\overset{\in\mathbb{R}}{m}\cdot\overset{\in\mathbb{R}^3}{\mathbf{v}_A})$$

$$F_B = \frac{d}{dt}(m\cdot\mathbf{v}_B) = m\frac{d}{dt}(\mathbf{v}_A + \overset{\in\mathbb{R}^3}{\mathbf{V}}) = m\left(\frac{d}{dt}\mathbf{v}_A + \underbrace{\frac{d}{dt}\mathbf{V}}_{0}\right) = F_A$$

using universal time

for any constant **V**

b)

$$\overset{\in\mathbb{R}^3}{f_A} = \overset{\in\mathbb{R}}{q}\overset{\in\mathbb{R}^3}{E_A} + q\overset{\in\mathbb{R}^3}{\mathbf{v}_A}\times\overset{\in\mathbb{R}^3}{B_A} \quad \text{(charge } q \text{ in frame A)}$$

$$f_B = qE_B + q(\mathbf{v}_A + \mathbf{v})\times B_B \quad \text{(charge } q \text{ in frame B)}$$

if Galilean then $f_A = f_B$ for any constant velocity **v** and for any difference between frames A & B and for any velocities $\mathbf{v}_A$!

$$\Rightarrow \quad \mathbf{E}_A + \mathbf{v}_A\times\mathbf{B}_A \equiv \mathbf{E}_B + \mathbf{v}_A\times\mathbf{B}_B + \mathbf{v}\times\mathbf{B}_B$$

$$\text{if } \mathbf{v}_A = 0 \Rightarrow \mathbf{E}_A = \mathbf{E}_B + \mathbf{v}\times\mathbf{B}_B$$

$$\text{if } \mathbf{v} = 0 \Rightarrow E_A = E_B \;\&\; B_A = B_B \text{ for any } \mathbf{v}_A$$

if $\mathbf{v}_A = 0$ (frame A at rest) then $\mathbf{E}_B$ field cannot be equal with $\mathbf{E}_A$.

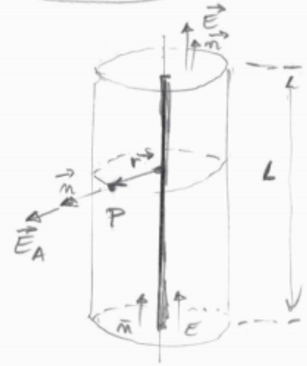
Example: The laboratory is the absolute reference. And the fields E_A & B_A are generated in the lab. The first probe A is fixed in the lab at a point x_0, y_0, z_0 ($v_A = 0$). The second one is moving through x_0, y, z_0 with speed v_B. Either the probes measure different forces or the fields E_B, B_B are different from E_A, B_A.

Notes 1.1

b) For the Lorentz force law in reference frame A : $\mathbf{f_A} = q\mathbf{E_A} + q\mathbf{v_A} \times \mathbf{B_A}$, the velocity $\mathbf{v_A}$ refers to the velocity of an idealized probe with charge a in the point A of the frame.

c) If the charge in the wire is at rest in frame A and frame B is moving along the wire with speed $|\mathbf{v}|$ then we have a current in the wire and thus a non null magnetic field $\mathbf{B}$.

Sol 1.1c



In a point P in the coordinates of the laboratory (at rest)

$f_A(P) = qE_A(P) + q\, v_A(P) \times B_A(P)$

Assuming charge is not moving!
Nothing in the frame A is moving!

We use 2 equations (Gauss)

$$\oiint_{\text{surface}} \underbrace{E_A \cdot n}_{\text{scalar product}} = \frac{1}{\varepsilon_0} \iiint_{\text{volume}} \text{charge}$$

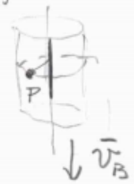
and $$\oint_{\text{curve}} B_A \cdot dr = \mu_0 I_A$$

Since in frame A charge is not moving there is no current $I_A = 0 \Rightarrow B_A = 0$ $\forall$ point P

E_A is constant in the lateral surface of the cylinder and parallel to the normal to it. The upper and lower surfaces of the cylinder cancel each other E_A (from symmetry):

$$\underbrace{2\pi r L}_{\text{lateral surface}} \cdot E_A(P) = \frac{1}{\varepsilon_0} \underbrace{\lambda L}_{\text{total charge}} \; ; \; E_A(r) = \frac{\lambda}{2\pi\varepsilon_0} \cdot \frac{1}{r}$$

In frame B the electric field is the same $E_B(r) = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}$



In the same point P (coordinates relative to a distant star) the charge is moving relative to the frame B and creates a current $I_B = \lambda \frac{\Delta L}{\Delta t} = \lambda v$

$B_B(r) = \mu_0 \lambda v \neq 0$

thus $f_A(P) \neq f_B(P)$!

The magnetic field of a wire is B= $\mu_0 I / (2 \pi R)$. The current intensity: 1 Ampere = 1 Coulomb/ 1 sec. 1 Coulomb =~ 6×10^{18} electrons. How do the electrons generate the magnetic Field? It could be an electron gun (not a wire) ! The `signal` speed of electrons in a metallic wire is about the speed of light (See electron speed). Electrons might be bound – lower speed!

P 1.2

Show that the force between two charges , each of one Coulomb separated by one kilometre, is equivalent to the weight of about one ton.

Sol P1.2

$$f = \frac{qQ}{4\pi\varepsilon_0}\frac{1}{r^2} \qquad \frac{1}{4\pi\varepsilon_0} \simeq 8.9\,10^9 N m^2 C^{-2}$$

$$f = 1\cancel{C}\cdot 1\cancel{C}\cdot 8.9\,10^9 \frac{N\cancel{m^2}}{\cancel{C^2}}\cdot\frac{1}{10^{2\times 3}\cancel{m^2}}$$

$$\approx 8.9\cdot 10^3 N$$

weight of a ton in gravitational field $g = 9.8\frac{m}{s^2}$

is $m\cdot g = 10^3 Kg \cdot 9.8\frac{m}{s^2} \approx 9.8\,10^3 N$

P 1.3

A joule is 1 kg m^2 s^{-2} and 1 erg is 1 g cm^2 s^{-2} .

a) How many ergs are there in one joule?

b) Since one erg is 1 statvolt statcoulomb and one joule is 1 volt C, use part a) to find the number of volts in a statvolt.

Sol P1.3

a) $1J = 1N\cdot 1m$ (definition $E = F\cdot$distance

$$= 1Kg\,\frac{1m}{1s^2}\cdot 1m = 1Kg\,\frac{m^2}{s^2}$$

$$1J = 1000g\; 100^2 cm\cdot s^{-2} = 10^{3+4} erg$$

b) $1J = 1V\cdot C$

$1VC = 10^7 erg = 10^7$ statvolt · statcoul

but $1C = c\cdot$statcoul $= \dot{3}\times 10^9$ statcoul

↑speed of light

$\Rightarrow 1V\cdot\dot{3}\cdot 10^9$ ~~statcoul~~ $= 10^7$ statvolt · ~~statcoul~~

$\Rightarrow \dot{3}\cdot 10^2 V = 1$ statcoul

P 1.4

Prove that any vector in the {e1,e2} plane anticommutes with the unit bivector e1e2, wheres any vector perpendicular to the plane commutes with this bivector.

Sol 1.4

a) $v \in \mathbb{R}^n \quad v \in \{e_1, e_2\} \Rightarrow e_1, e_2 \in \mathbb{R}^n$

$v = \alpha^1 e_1 + \alpha^2 e_2 \quad \forall \alpha^1, \alpha^2 \in \mathbb{R}$ 1,2 indices

$$e_1, e_2 \text{ base} \Rightarrow \begin{cases} e_1^2 = 1,\ e_2^2 = 1 \ \&\ e_1 e_2 = -e_2 e_1 & \text{type space} \\ \text{or} \\ e_1^2 = -1,\ e_2^2 = 1 \ \&\ e_1 e_2 = -e_2 e_1 & \text{type time} \end{cases}$$

$$\left.\begin{array}{l} e_1 e_2 (\alpha^1 e_1 + \alpha^2 e_2) = -\alpha^1 e_2 + \alpha^2 e_1 \\ (\alpha^1 e_1 + \alpha^2 e_2) e_1 e_2 = \alpha^1 e_2 - \alpha^2 e_1 \end{array}\right\} \Rightarrow e_1 e_2 v = -v e_1 e_2$$

or

$$\left.\begin{array}{l} \overbrace{e_1 e_2 (\alpha^1 e_1}^{-1} + \alpha^2 e_2) = \alpha^1 e_2 + \alpha^2 e_1 \\ (\alpha^1 e_1 + \alpha^2 e_2) e_1 e_2 = -\alpha^1 e_2 - \alpha^2 e_1 \end{array}\right\} \Rightarrow e_1 e_2 v = -v e_1 e_2$$

regardless if e_1, e_2 are of type space or time

b) by definition $w \perp v$ if $wv = -vw$
(where v, w are vectors in $\mathbb{R}^n$)
In this case $v = \alpha^1 e_1 + \alpha^2 e_2 \ \forall \alpha^1, \alpha^2 \in \mathbb{R}$ 1,2: indices
$\Rightarrow e_1 \perp w \ \&\ e_2 \perp w \Rightarrow e_1 w = -w e_1 \ \&\ e_2 w = -w e_2$

$$e_1 e_2 w = e_1 (e_2 w) = -e_1 w e_2 = -(e_1 w) e_2 = w e_1 e_2$$

Note that for $v, w \in \mathbb{R}^3$ there is just one direction that is $\perp$ to $\{v_1, v_2\}$.
For $v, w \in \mathbb{R}^n$ we have many directions $\perp$ to $\{v_1, v_2\}$.
For example $v_4 \perp \{v_1, v_2\}$; $v_5 \perp \{v_1, v_2\}$ ...

P 1.5

A corner cube is constructed by placing reflecting coatings on the interior surfaces of the three planes and meet at the corner of the cube (the other half of the cube is removed). Use the Pauli algebra to show that any ray of light incident on the corner of the cube and bouncing on all three surfaces will be inverted, that is emerges in the precise direction from which it came.

Sol 1.5 Pauli algebra elements are the bivectors: $1, (e_1e_2), (e_2e_3), (e_3e_1)$ $\quad (e_i \in \mathbb{R}^3 \; i=1..3)$

since
$$(e_1e_2)(e_2e_3) = +e_3e_1 \quad \& \quad (e_1e_2)(e_1e_2) = -e_1e_1e_2e_2 = -1$$
$$(e_2e_3)(e_3e_1) = -(e_1e_2) \quad \& \quad (e_2e_3)(e_2e_3) = -1$$
$$(e_3e_1)(e_1e_2) = -(e_2e_3) \quad \& \quad (e_3e_1)(e_3e_1) = -1$$

Let r be a vector in $\mathbb{R}^3$ then we show that $(e_1e_2)\, r\, (e_1e_2)$ is the reflection of r in the plane $\{e_1e_2\}$

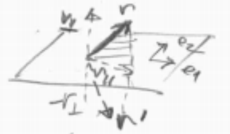
In $\mathbb{R}^3$ the vector r can be decomposed in 2 parts: one $r_\parallel$ that is in the plane $\{e_1e_2\}$ and one that is perpendicular to it $r_\perp$. Thus $r = r_\perp + r_\parallel$ and $r_\parallel = \alpha^1 e_1 + \alpha^2 e_2$

$$(e_1e_2)\, r_\parallel\, (e_1e_2) = (e_1e_2)(\alpha^1 e_1 + \alpha^2 e_2)(e_1e_2) = \alpha^1 e_1 + \alpha^2 e_2 = r_\parallel$$

Since $r_\perp$ must be perpendicular to any vector in $\{e_1, e_2\}$

$\Rightarrow r_\perp \perp e_1 \;\&\; r_\perp \perp e_2 \Rightarrow r_\perp e_1 = -e_1 r_\perp \;\&\; r_\perp e_2 = -e_2 r_\perp$

$(e_1e_2)\, r_\perp\, (e_1e_2) = -e_1e_2 r_\perp e_2 e_1 = +e_1 r_\perp e_1 = -r_\perp$: a reflection



Similar for (e_2e_3) & (e_3e_1)

$r' = (e_1e_2)\, r\, (e_1e_2)$ is a reflection on $\{e_1, e_2\}$

In the cube corner the reflection is applied on all 3 planes ← regardless of the order!!!

eg. $(e_3e_1)\Big((e_2e_3)\big(\underbrace{e_1e_2\, r\, e_1e_2}\big)(e_2e_3)\Big)e_3e_1\Big) =$

$= e_3e_1e_2e_3e_1e_2\, r = e_3e_1e_3e_1\, r = -r$ thus reverses sign.

Alternative solution using Hestenes's approach in $\mathbb{R}^n$

If a, r are vectors then $-ara$ is the reflection of a in the plane perpendicular to a

In the case of the cube we have 3 normal vectors $e_1, e_2, e_3 \Rightarrow$

$$r' = (-1)^3 e_1 e_2 e_3\, r\, e_3 e_2 e_1 \quad \text{and } r = \alpha^1 e_1 + \alpha^2 e_2 + \alpha^3 e_3$$

since $(-1)\, e_1e_2e_3\, \alpha^1 e_1\, e_3 e_2 e_1 = -\alpha_1 e_1$ & similar for e_2, e_3

it results $r' = -r$ (regardless of order of reflections)

$$e_2e_3e_1e_1e_1e_3e_2 = -e_1 \ldots$$

P 1.6

Show that paravector involutions defined by spatial reversal ($p_0+\mathbf{p} \rightarrow p_0-\mathbf{p}$), and by complex conjugation ($p^\mu e_\mu \rightarrow p^{\mu *} e_\mu$), when applied to products, do require a reversal in the order of multiplications.

Sol 1.6

a) let $p, q \in \mathbb{R}^n$ pure vectors we will show that $\overline{pq} = \bar{q}\bar{p}$

$p = \sum_i \alpha_i e_i \quad (\alpha_i \in \mathbb{R}) \qquad q = \sum_j \beta_j e_j \quad (\beta_j \in \mathbb{R})$ $\quad e_i, e_j$ orthonormal base, no scalars

$$\overline{pq} = \overline{\left(\sum \alpha_i e_i\right)\left(\sum \beta_j e_j\right)} = \overline{\sum_k \alpha_k \beta_k + \sum_{\substack{i,j \\ i<j}} (\alpha_i \beta_j - \alpha_j \beta_i) e_i e_j}$$

$$= \sum_k \alpha_k \beta_k - \sum_{i<j} (\alpha_i \beta_j - \alpha_j \beta_i) e_i e_j$$

$$\bar{q}\bar{p} = \left(\overline{\sum_j \beta_j e_j}\right)\left(\overline{\sum_i \alpha_i e_i}\right) = \left(\sum_j \beta_j e_j\right)\left(\sum \alpha_i e_i\right) =$$

$$= \sum_k \alpha_k \beta_k + \sum_{i<j} (\alpha_i \beta_j - \alpha_j \beta_i) e_j e_i = \overline{pq}$$

for the paravectors:

$p = p_0 + \mathbf{p} \qquad q = q_0 + \mathbf{q}$

$$\overline{pq} = \overline{(p_0+\mathbf{p})(q_0+\mathbf{q})} = \overline{p_0 q_0 + q_0 \mathbf{p} + p_0 \mathbf{q} + \mathbf{pq}} = p_0 q_0 - q_0 \mathbf{p} - p_0 \mathbf{q} + \overline{\mathbf{pq}}$$

$$\bar{q}\bar{p} = (q_0 - \mathbf{q})(p_0 - \mathbf{p}) = p_0 q_0 - q_0 \mathbf{p} - p_0 \mathbf{q} + \mathbf{qp}$$

b) $p = \sum_\mu p_\mu e_\mu \quad (p_\mu \in \mathbb{C}) \qquad p^\dagger = \sum_\mu p_\mu^* e_\mu$ $\quad$ where $*$ is complex conjuga

$$q = \sum_\lambda q_\lambda e_\lambda \qquad q^\dagger = \sum_\lambda q_\lambda^* e_\lambda$$

$$(pq)^\dagger = \left(\sum_\mu \sum_\lambda p_\mu q_\lambda e_\mu e_\lambda\right)^\dagger = \sum_\mu \sum_\lambda (p_\mu q_\lambda)^* e_\mu e_\lambda$$

$$q^\dagger p^\dagger = \left(\sum_\lambda q_\lambda^* e_\lambda\right)\left(\sum_\mu p_\mu^* e_\mu\right) = \sum_\mu \sum_\lambda (p_\mu q_\lambda)^* e_\lambda e_\mu \quad !?$$

Lets use $\mathbb{R}^2$

$$\overline{(p_1 e_1 + p_2 e_2)(q_1 e_1 + q_2 e_2)} = (p_1 q_1 + p_2 q_2)^* + (p_1^* q_2^* - p_2^* q_1^*) e_1 e_2$$

$$(q_1^* e_1 + q_2^* e_2)(p_1^* e_1 + p_2^* e_2) = p_1^* q_1^* + p_2^* q_2^* + (q_1^* p_2^* - p_1^* q_2^*) e_1 e_2$$

P 1.7

Prove explicitly that the canonical element $e_1e_2e_3$ in the Pauli algebra squares to -1 and that it commutes with all the elements of the algebra. Extend the result to n-dimensional Euclidian space by calculating the square of the canonical element and determining its commutator with vectors in the more general case.

Sol 1.7 in Pauli algebra $j = e_1e_2e_3$ $e_1 = \sigma_1$, $e_2 = \sigma_2$, $e_3 = \sigma_3$

$i = 1..3$, e_i orthonormal base in $\mathbb{R}^3$; elements of algeba e_1e_2, e_2e_3, e_3e_1

$$j^2 = (e_1e_2e_3)(e_1e_2e_3) = e_1e_2e_3e_1e_2e_3 = e_2e_3e_2e_3 = -1$$

$je_1 = e_1e_1e_2e_3 = (e_1e_2e_3)e_1 = e_1 j$; $je_2 = e_1e_2e_3e_2 = e_2 j$; $je_3 = e_1e_2e_3e_3 = e_3 j$

$$j(e_1e_2) = e_1e_2e_3 \cdot e_1e_2 = e_1e_2(e_1e_2e_3) = e_1e_2 j$$

$$j(e_2e_3) = e_1e_2e_3e_2e_3 = e_2e_3(e_1e_2e_3) = e_2e_3 j$$

$$j(e_3e_1) = e_1e_2e_3e_3e_1 = -e_1e_3e_3e_1e_2 = -e_1e_3(e_1e_2e_3) = e_3e_1 j$$

— for $\mathbb{R}^n$

$j = e_1e_2 \dots e_n$ $j^2 = e_1e_2 \dots e_n e_1e_2 \dots e_n$ j: pseudoscalar

$(n-1)$ inversion $+ (n-2) + \dots 1$ } total $= \frac{n(n-1)}{2}$ inversions

$$j^2 = (-1)^{\frac{n(n-1)}{2}}$$

$$\Rightarrow j^2 = \begin{cases} (-1) & n = 2, 6, 10 \\ (-1) & n = 3, 7, \dots \\ (+1) & n = 4, 8, \dots \\ (+1) & n = 5, 9, \dots \end{cases}$$

if $e_k^2 = 1$ for $\forall k$

for a commutator a: $ja = aj$

if $a = e_k$ is a basis item $(e_1e_2 \dots e_{k-1}e_ke_{k+1} \dots e_n)e_k = (-1)^{n-1} e_k(e_1 \dots e_n)$ [$(-1)^{n-k}$, $(-1)^{k-1}$]

if a is a bivector $(e_1e_2 \dots e_n)e_ke_m = (-1)^{2(n-1)} e_ke_m(e_1e_2 \dots e_n)$

for a p-vector $j \cdot a = (-1)^{p(n-1)} a \cdot j$ n odd $ja = aj$

for n even depends on the "arity" of the p.

P 1.8

Use equations **a b** = (**a b** + **b a**)/2 + (**a b – b a**)/2 and **a** ∧ **b** = *i* **a** × **b** to show that the product of two unit spatial vectors is a rotational element: $\hat{\mathbf{a}}\,\hat{\mathbf{b}} = \exp(i\,\theta)$ where $i\,\theta$ is a vector representing the plane containing $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, and θ is the angle that rotates $\hat{\mathbf{a}}$ into $\hat{\mathbf{b}}$.

Sol 1.8

$\hat{a}, \hat{b}$ unit spatial vectors $\Rightarrow$ $\hat{a}^2 = 1$ & $\hat{b}^2 = 1$

and since i is defined here as $i = e_1 e_2 e_3$

we assume $\hat{a} \in \mathbb{R}^3$ & $\hat{b} \in \mathbb{R}^3$

$\hat{a}^2 = 1 \Rightarrow |\hat{a}| = 1$; $\hat{b}^2 = 1 \Rightarrow |\hat{b}| = 1$

$i^2 = (e_1 e_2 e_3)(e_1 e_2 e_3) = e_2 e_3 e_2 e_3 = -1$

$\hat{a}\hat{b} = \underbrace{(\hat{a} \cdot \hat{b})}_{\text{symmetric}} + \underbrace{(\hat{a} \wedge \hat{b})}_{\text{antisimmetric}}$ (axial vector!)

$(\hat{a} \cdot \hat{b}) = |\hat{a}||\hat{b}| \cos\theta \quad (\hat{a} \wedge \hat{b}) = i\,\hat{a} \times \hat{b} = i\,|\hat{a}||\hat{b}| \sin\theta$

where θ is the angle from $\hat{a}$ into $\hat{b}$

The new base is $\{1, e_1, e_2, e_3\}$ and the result is a complex paravector $\hat{a}\hat{b} = \cos\theta + i \sin\theta$

since $e_1 \times e_2 = e_3$ and $(e_1 e_2 e_3) e_1 e_2 = e_3$,

$e_2 \times e_3 = e_1$ and $(e_1 e_2 e_3) e_2 e_3 = e_1$

$e_3 \times e_1 = e_2$ and $(\underbrace{e_1 e_2 e_3}_{i}) e_3 e_1 = e_2$

and $\hat{a}\hat{b}$ is a paravector in any base e_1, e_2, e_3,

$\hat{a} = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \quad \hat{b} = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3$

$$\hat{a}\hat{b} = \underbrace{(\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3)}_{\text{scalar}} + (\alpha_1\beta_2 - \alpha_2\beta_1)\underbrace{e_1 e_2}_{i e_3} + (\alpha_2\beta_3 - \alpha_3\beta_2)\underbrace{e_2 e_3}_{i e_1} + (\alpha_3\beta_1 - \alpha_1\beta_3)\underbrace{e_3 e_1}_{i e_2}$$

P 1.9

Noncommuting rotations. Consider the rotation elements $R_1 = \exp(\mathbf{e_2 e_3}\,\pi/2)$ and $R_2 = \exp(\mathbf{e_3 e_1}\,\pi/2)$, which can be used in transformations of the form RvR^{-1} to rotate vectors by 180° in the $\{e_2,e_3\}$ plane and the $\{e_3,e_1\}$ plane, respectively.

a) Prove explicitly that $R_1R_2 = -R_2R_1$, and that this product represents a 180° rotation in the $\{e_2,e_1\}$ plane.

b) Show that the element $\exp(\mathbf{e_2e_3}\,\pi/2 + \mathbf{e_3e_1}\,\pi/2)$ also represents a rotation, but that both the magnitude and the plane of rotation are distinct from those of R_1R_2 and R_2R_1 .

Sol 1.9

a) $R_1 = \exp(e_2e_3\frac{\pi}{2}) = \cos\frac{\pi}{2} + \sin\frac{\pi}{2}e_2e_3 = e_2e_3$

$R_2 = \exp(e_3e_1\frac{\pi}{2}) = \cos\frac{\pi}{2} + \sin\frac{\pi}{2}e_3e_1 = e_3e_1$

$R_1R_2 = (e_2e_3)(e_3e_1) = e_2e_1 = \cos\frac{\pi}{2} + \sin\frac{\pi}{2}e_2e_1 = \exp(e_2e_1\frac{\pi}{2})$

$R_2R_1 = (e_3e_1)(e_2e_3) = e_3e_1e_2e_3 = e_1e_2 = -e_2e_1 = -R_1R_2$

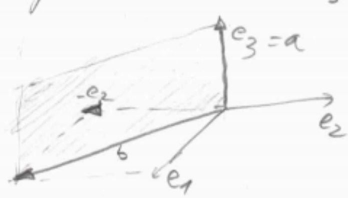
b) $\exp(e_2e_3\,\pi/2 + e_3e_1\frac{\pi}{2}) =$

$= \exp\left((e_2e_3 + e_3e_1)\frac{\pi}{2}\right) =$

$= \exp\left(\underbrace{e_3}_{a}\underbrace{(e_1-e_2)}_{b}\frac{\pi}{2}\right)$

$\|a\| = 1 \qquad \|b\| = \|(e_1-e_2)(e_1-e_2)\| = \|e_1^2+e_2^2\| = \|2\| = \sqrt{2}$

The plane of rotation is $e_3 \perp (e_1-e_2)$



P 1.11

Projectors. Let P=(1+**n**)/2 where **n** is a real unit vector. The element P is called a projector. (P is the conjugate (1-**n**)/2). Prove the properties:

a) $P^2 = P = P^+$ (idempotent, real)

b) PP = PP = 0 (null, self-orthogonal)

c) P + P =1 (complete)

Which of these properties are incompatible with division algebra (all non zero element is invertible)?

Sol 1.11

$n \in \mathbb{R}^k \quad n^2 = 1 \quad \& \quad P = \frac{1}{2}(1+n) \quad \bar{P} = \frac{1}{2} - \frac{1}{2}n$

a) $P^2 = \frac{1}{2}(1+n)\frac{1}{2}(1+n) = \frac{1}{4}(1+2n+n^2) = \frac{1}{4}2(1+n) = P$

(idempotent) so $P^2 = P$

b) $\bar{P} = \frac{1}{2}(1-n)$

$P\bar{P} = \frac{1}{2}(1+n)\frac{1}{2}(1-n) = \frac{1}{4}(1-n^2) = 0$

$\bar{P}P = \frac{1}{2}(1-n)\frac{1}{2}(1+n) = \frac{1}{4}(1-n^2) = 0$

(selforthogonal)

c) $P + \bar{P} = \frac{1}{2}(1+n) + \frac{1}{2}(1-n) = \frac{1}{2}2 = 1$

(complete)

Both P and $\bar{P}$ are divisors of 0 from b) selforthogonal

P 1.12

Spectral decomposition. Let x= x^0 + **x** be a real paravector.

a) Find the projector P of the form (1+**n**)/2 ; $\mathbf{n}^2=1$ such that x = $\lambda_+P + \lambda_-P$; where λ_+, λ_- are scalars.

b) Find the 'eigenvalues' λ_+, λ_-

c) From the properties next show that any power x^n of x can be written: $x^n = \lambda_+^n P + \lambda_-^n P$;

d) Let f(x) be an analytic function of its argument, with a series expansion in powers xn. Prove the relation: $f(x)=f(\lambda_+)P + f(\lambda_-)P$. (This relation can be used to define even non-analytical functions of paravectors)

Sol 1.12

a) $$x_0 + \mathbf{x} = \lambda_+(1+n)/2 + \lambda_-(1-n)/2 = \frac{(\lambda_+ + \lambda_-)}{2} + \frac{(\lambda_+ - \lambda_-)}{2} n$$

$$\Rightarrow x_0 = (\lambda_+ + \lambda_-)/2 \; ; \; n = \frac{2\mathbf{x}}{(\lambda_+ - \lambda_-)} \; ; \; n^2 = \underbrace{\frac{4x^2}{(\lambda_+ - \lambda_-)^2}}_{\in \mathbb{R}} = 1$$

b) $$\begin{cases} x_0 = (\lambda_+ + \lambda_-)/2 & \Rightarrow 4x_0^2 = \lambda_+^2 + 2\lambda_+\lambda_- + \lambda_-^2 \\ 4x^2 = (\lambda_+^2 - 2\lambda_+\lambda_- + \lambda_-^2) & \end{cases}$$

$$\Rightarrow 4(x_0^2 - x^2) = 4\lambda_+\lambda_-$$

$$\Rightarrow \begin{cases} \lambda_+ + \lambda_- = 2x_0 \\ \lambda_+\lambda_- = x_0^2 - x^2 \end{cases} \Rightarrow \begin{matrix} \lambda_+ = x_0 + \sqrt{x^2} \\ \lambda_- = x_0 - \sqrt{x^2} \end{matrix} \quad (\sqrt{x^2} = \|x\|)$$

$$P = \left(1 + \frac{\mathbf{x}}{\|x\|}\right)/2 \; ; \; x_0 + \mathbf{x} = (x_0 + \|x\|)\left(1 + \frac{\mathbf{x}}{\|x\|}\right)/2 + (x_0 - \|x\|)\left(1 - \frac{x}{\|x\|}\right)$$

$$\bar{P} = \left(1 - \frac{x}{\|x\|}\right)/2$$

c) $$(x_0 + \mathbf{x})^2 = (x_0 + \mathbf{x})(x_0 + \mathbf{x}) = (\lambda_+ P + \lambda_- \bar{P})(\lambda_+ P + \lambda_- \bar{P}) = \lambda_+^2 \underbrace{P^2}_{P} + \lambda_+\lambda_- \underbrace{P\bar{P}}_{0} + \lambda_+\lambda_- \underbrace{\bar{P}P}_{0} + \lambda_-^2 \underbrace{\bar{P}^2}_{\bar{P}} =$$

$$= \lambda_+^2 P + \lambda_-^2 \bar{P}$$

$$(x_0 + \mathbf{x})^n = (x_0 + x)^{n-1}(x_0 + \mathbf{x}) = (\lambda_+^{n-1} P + \lambda_-^{n-1} \bar{P})(\lambda_+ P + \lambda_- P) = \lambda_+^n P + \lambda_-^n \bar{P}$$

d) $f(x) = \sum_i \alpha_i x^i$ x paravector $\Rightarrow x = x_0 + \mathbf{x} = \lambda_+ P + \lambda_- \bar{P}$

$$f(x) = \sum_i \alpha_i (\lambda_+ P + \lambda_- \bar{P})^i = \sum_i \alpha^i (\lambda_+^i P + \lambda_-^i \bar{P}) = f(\lambda_+)P + f(\lambda_-)\bar{P}$$

P 1.13

Rotating frame. Let **r** be a time dependent position in a frame rotating at an angular velocity **ω** about an axis through the origin **r**=0. The same position in the stationary frame is given by

$$\mathbf{r}' = R\mathbf{r}R^{-1}$$

where R= exp(-*i***ω** t/2). [Note that **ω** is usually treated as a vector pointing along the axis of rotation. This is satisfactory in three-dimensional space, where there exists a unique axis for any rotation, but in spaces of higher dimensions, one should instead represent the angular velocity by a bivector in the plane of rotation. In three dimensions, the bivector is *i***ω** .] Take the derivatives with respect to time t to relate the velocity and acceleration of the stationary frame to the position, velocity and acceleration in the rotating frame. In particular, prove:

$$\ddot{r}' = R\left[\ddot{r} + 2\,\boldsymbol{\omega} \times \dot{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})\right]R^{-1}$$

(two solutions on the next 2 pages)

(Sol 1.13a) In general, for 2 vectors a, b we have $a \wedge b = \frac{ab - ba}{2}$. We consider now $a, b \in R^3$ (and Cl_3 algebra) where $a \wedge b = i\, a \times b$; $i^2 = -1$

The rules for derivative of multiplication are the same: $(\dot{fg}) = \dot{f}g + \dot{g}f$ for any functions of time; where f, g can be multivector functions of time.

Formaly we have $\dot{R} = (\dot{\exp(-\frac{i\omega t}{2})}) = -\frac{i\omega}{2} R$ where i is the pseudoscalar that commutes with any vector $\mathbf{w}$. We can also use the bivector of the plane of rotation as an equivalent:

$R = \cos\frac{wt}{2} + \sin\frac{wt}{2}\, e_1 e_2$; $R^{-1} = \cos(-\frac{wt}{2}) + \sin(-\frac{wt}{2})\, e_1 e_2$; where $e_1 e_2$ is an orthonormal base of the plane of rotation and w is a constant scalar (the angular speed)

$$\dot{R} = \frac{w}{2}\left(-\sin\frac{wt}{2} + \cos\frac{wt}{2}\, e_1 e_2\right) =$$

$$= \frac{w}{2} e_1 e_2 \left(\cos\frac{wt}{2} + \sin\frac{wt}{2}\, e_1 e_2\right) = \frac{w}{2} e_1 e_2 R$$

since $e_1 e_2 e_1 e_2 = -1$

By definition $\underset{\in R}{\frac{w}{2}} e_1 e_2 = -\frac{i\mathbf{w}}{2} \in C$. And $\dot{R}^{-1} = \frac{i\mathbf{w}}{2} R^{-1}$

(an abuse of notation for w)

$$\dot{r}' = (\dot{R r R^{-1}}) = \dot{R} r R^{-1} + R \dot{r} R^{-1} + R r \dot{R}^{-1} = -\frac{i\mathbf{w}}{2} R r R^{-1} + R \dot{r} R^{-1} + R r \frac{i\mathbf{w}}{2} R^{-1}$$

$$= R\left[\dot{r} + \frac{-i\mathbf{w} r + r i\mathbf{w}}{2}\right] R^{-1} = R[\dot{r} + w \times r] R^{-1}$$

since $r \wedge i w = \mathbf{w} \times r$ (?!)

$$\ddot{r}' = \dot{R}[\dot{r} + w\times r] R^{-1} + R[\ddot{r} + w \times \dot{r}] R^{-1} + R[\dot{r} + w \times r] \dot{R}^{-1} =$$

$$= -\frac{i w}{2} R[\dot{r} + w \times r] R^{-1} + R[\ddot{r} + w \times \dot{r}] R^{-1} + R[\dot{r} + w \times r] \frac{i w}{2} R^{-1}$$

$$= R\left[\underbrace{-\frac{i w}{2}\dot{r} + \dot{r}\frac{i w}{2}} \underbrace{- \frac{i w}{2}(w \times r) + (w \times r)\frac{i w}{2}} + \ddot{r} + w \times \dot{r}\right] R^{-1}$$

$$= R[\; w \times \dot{r} + w \times (w \times r) + \ddot{r} + w \times \dot{r}] R^{-1} = R[\ddot{r} + 2 w \times r + w \times (w \times r)] R$$

(Sol 1.3b) We will use the (iso)morphism between Cl_3 on $\mathbb{R}$ and the space of paravectors with coefs in $\mathbb{C}$

$$\overset{\in\mathbb{R}}{s}+\sum_{k=1}^{3}\overset{\in\mathbb{R}}{\alpha_k} e_k + \overset{\in\mathbb{R}}{\beta_3} e_1e_2 + \overset{\in\mathbb{R}}{\beta_1} e_2e_3 + \overset{\in\mathbb{R}}{\beta_2} e_3e_1 + \overset{\in\mathbb{R}}{\mu} e_1e_2e_3 \longleftrightarrow$$

$$\overset{\in\mathbb{C}}{(s+i\mu)} + \sum_{k=1}^{3}\overset{\in\mathbb{C}}{(\alpha_k+i\beta_k)} e_k$$

The cross (a vectorial) product $\times$ is defined as having distributivity property and for each pair of orthonormal base elements of the Paravectors $\mathcal{P}$ over $\mathbb{C}$

$$e_1\times e_2 = e_3\ ,\quad e_2\times e_3 = e_1\ ,\quad e_3\times e_1 = e_2\ ;\quad e_i\times e_j = -e_j\times e_i$$

and distributivity $a\times(b+c) = a\times b + a\times c$ $\quad a,b,c\in\mathcal{P}$ over $\mathbb{C}$

It is easy to see that if $a=\sum_{k=1}^{3}\overset{\in\mathbb{R}}{\alpha_k} e_k$ and $b=\sum_{k=1}^{3}\overset{\in\mathbb{R}}{\beta_k} e_k$ $\quad a,b\in Cl_3$

that $a\wedge(b+c) = a\wedge b + a\wedge c$ and $\overset{\in\mathbb{R}}{\alpha}(a\wedge b) = (\alpha a)\wedge b = a\wedge(\alpha\beta)$

and any property of $\wedge$ applied to e_k, e_m will appear on the general vectors composed of the basis.

Thus $\quad e_1\wedge e_2 = \dfrac{e_1e_2-e_2e_1}{2} = e_1e_2 \longleftrightarrow ie_3 = i(e_1\times e_2)$

see the (iso)morphism

$$e_2\wedge e_3 = e_2e_3 \longleftrightarrow ie_1 = i(e_2\times e_3)$$
$$e_3\wedge e_1 = e_3e_1 \longleftrightarrow ie_2 = i(e_3\times e_1)$$

and as a consequence $\quad a\wedge b \longleftrightarrow i(a\times b)$

$Cl_3 \ni R = \cos(\frac{\Omega t}{2}) + \sin(\frac{\Omega t}{2})e_1e_2 \longleftrightarrow \exp(\frac{i\mathbf{w}t}{2})$ $\in\mathcal{P}$ over $\mathbb{C}$ where $\mathbf{w}\perp\{e_1,e_2\}$

$\dot R = \frac{\Omega}{2}e_1e_2R$; $\{e_1,e_2\}$ rotation plane $\longleftrightarrow \frac{i\mathbf{w}}{2}R$

$$\dot r' = (\dot{\overline{RrR^{-1}}}) = \dot R r R^{-1} + R\dot r R^{-1} + Rr\dot R^{-1} = R\left[\tfrac{\Omega}{2}e_1e_2 r + \dot r - \tfrac{r\Omega e_1e_2}{2}\right]R^{-1} =$$
$$= R\left[\dot r + \Omega\,\frac{(e_1e_2r - re_1e_2)}{2}\right]R^{-1} = R[\dot r + \Omega(e_1e_2)\wedge r]R^{-1} \longleftrightarrow R[\dot r + i\mathbf{w}\times r]R^{-1}$$

$$\ddot r' = \dot{\overline{R[\dot r + \tfrac{\Omega}{2}(e_1e_2r - re_1e_2)]R^{-1}}} = \dot R[\quad]R^{-1} + R[\ddot r + \tfrac{\Omega}{2}(e_1e_2\dot r - \dot re_1e_2)]R^{-1} +$$
$$+ R[\quad]\dot R^{-1} =$$
$$= R\left[\tfrac{\Omega}{2}e_1e_2(\dot r + \Omega(e_1e_2)\wedge r) + \ddot r + \Omega(e_1e_2)\wedge\dot r - (\dot r + \Omega(e_1e_2)\wedge r)\tfrac{\Omega}{2}e_1e_2\right]R^{-1}$$
$$= R\Big[\ddot r + \underbrace{\tfrac{\Omega}{2}(e_1e_2\dot r - \dot re_1e_2) + \Omega(e_1e_2)\wedge\dot r}_{\substack{2\Omega(e_1e_2)\wedge\dot r\\ \sim\, 2i\mathbf{w}\times\dot r}} + \underbrace{\tfrac{\Omega}{2}\big(e_1e_2(\Omega(e_1e_2)\wedge r) - \Omega((e_1e_2)\wedge r)e_1e_2\big)}_{\substack{\Omega(e_1e_2)\wedge[\Omega(e_1e_2)\wedge r]\\ \sim\, i\mathbf{w}\times(i\mathbf{w}\times r)}}\Big]R^{-1}$$

P 1.14

Antisymetric products. It is often convenient to consider antisymetric ("outer") products of the paravector basis elements e_μ , μ=0,1,2,3 of $\mathcal{C}\ell_3$ and their spatial reversals. Such products, indicated by ∧ (...), are defined to change the sign under the interchange of any two indices, thus

$\wedge < ...e_\mu\bar{e}_\nu ...> = - \wedge < ...e_\nu \bar{e}_\mu ...>$. All permutation of the indices can be generated by successive applications of such interchanges. Prove that:

$\wedge < e_\mu\bar{e}_\nu > = < e_\mu\bar{e}_\nu >_{\mathscr{V}}$ where $\mathscr{V}$ is the Vector part

$\wedge < e_\lambda \bar{e}_\mu e_\nu > = < e_\lambda \bar{e}_\mu e_\nu >_{\mathscr{I}}$ where $\mathscr{I}$ is the Imaginary part

Show that if the basis elements are all spatial vectors i $<e_je_ke_l> = \varepsilon_{jkl}$, the fully antisymetric (levi-Cevita) symbol in three dimensions, defined by

$\varepsilon_{123}= \varepsilon_{231}= \varepsilon_{312}= -\varepsilon_{213}= - \varepsilon_{132} = - \varepsilon_{321} = -1.$

Also note, but do not prove (too tedious!) the reltions:

and is the fully antisymetric (Levi-Cevita) symbol in four dimensions with $\varepsilon_{1230} =1$.

(skipped the proof for now)

P 1.15

Golden ratio an five-pointed stars. Let R be an invertible rotation operator of the form $\exp(\mathbf{e}_1\mathbf{e}_2\theta) = \cos\theta + (\mathbf{e}_1\mathbf{e}_2)\sin\theta$, with θ chosen such that $R^5 - 1 = 0$.

a) Factor the expression $R^5 - 1 = 0$ to show that unless R is identitity, it must satisfy:

$R^{-2} + R^{-1} + 1 + R + R^2 = 0$ and solve this equation to obtain exact numerical expression for cos θ. Thre should be four distinct values of θ in the range $-\pi < \theta < \pi$ that satisfy the above expression.

b) Let **v** be any vector in the e1e2 plane and interpret the expression:

$\mathbf{v}(R^{-2} + R^{-1} + 1 + R + R^2)$

as stating the existence of a closed five-sided polygon. Sketch the plygon, showing the directions of the vectors that make up the sides, for each of the four allowed angles θ.

c) Prove that the ratio of the distance between nonadjiacent vertices of the plygon to the length of a side is ether the golden ratio g or its inverse, namely $g^{\pm 1} = 1/2(\sqrt{5} \pm 1)$.

Sol. 1.15

a) It's easy to show that $R \cdot R^m = R^m R$ with $\forall m \in \mathbb{Z}$

$R^m = \cos(m\theta) + e_1e_2 \sin(m\theta)$ $\Rightarrow (R-1)(R^4+R^3+R^2+R+1) = R^5-1=0$

$\Rightarrow$ since $R \neq 1$ $R^4+R^3+R^2+R+1 = 0 \Rightarrow$

$R^{-2}(R^4+R^3+R^2+R+1) = 0 \Rightarrow R^2+R+1+R^{-1}+R^{-2} = 0$

Since both the scalar and the coef of the bivector must be zero $\Rightarrow \cos 2\theta + \cos\theta + 1 + \cos(-\theta) + \cos(-2\theta) = 0$

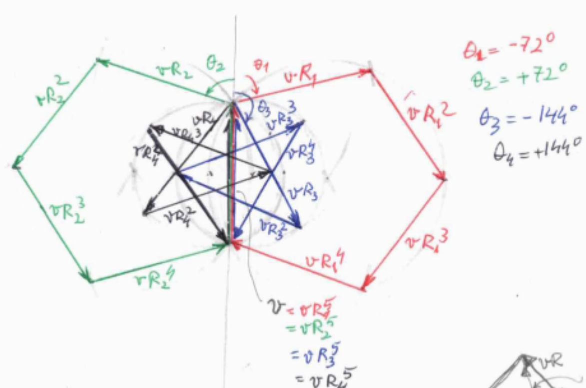
$\Rightarrow 2\cos 2\theta + 2\cos\theta + 1 = 0 \Rightarrow 2(2x^2-1) + 2x + 1 = 0$

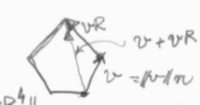
where $x = \cos\theta$; with solutions $x_{1/2} = (-1 \pm \sqrt{5})/4$

both solutions are valid $|x_{1,2}| < 1$. Since $\cos\theta = \cos(-\theta)$

we have 4 solutions in the interval $(-\pi, \pi)$

b)





c) distance between 2 nonadjacent vertices :

$\|v + vR\|$, $\|vR + vR^2\|$, $\|vR^2 + vR^3\|$, $\|vR^3 + vR^4\|$

we know that $\|R\| = 1$ (by definition & $\cos^2\theta + \sin^2\theta = 1$)

Using associativity $\|v + vR\| = \|(v+vR)R\| = \|(v+vR)R^2\| \ldots$

We are interested in the ratio $r = \frac{\|v + vR\|}{\|v\|}$. We can rescale by imposing $\|v\| = 1$ (using $n = \frac{v}{\|v\|}$ for example); n in the $\{e_1 e_2\}$ plane

$n = \cos\alpha\, e_1 + \sin\alpha\, e_2 \Rightarrow nR = \cos(\alpha+\theta)e_1 + \sin(\alpha+\theta)e_2 \Rightarrow$

$r^2 = \|n + nR\|^2 = \cos^2\alpha + \sin^2\alpha + \cos^2(\alpha+\theta) + \sin^2(\alpha+\theta) + 2\cos\alpha\cos(\alpha+\theta) + 2\sin\alpha\sin(\alpha+\theta)$

$= 2 + 2\cos\theta$ where θ is from a) above

$= 2 + \frac{-1 \pm \sqrt{5}}{2} = \frac{3 \pm \sqrt{5}}{2}$

$g^2 = \left(\frac{1}{2}(\sqrt{5} \pm 1)\right)^2 = \frac{1}{4}(5 + 1 \pm 2\sqrt{5}) = \frac{3 \pm \sqrt{5}}{2}$

$\Big\}\ r^2 = g^2 \Rightarrow r = g$

P 1.16

Fibonacci sequence. Define a parameter γ by $\gamma = gP - g^{-1}\bar{P}$, where P is the projector $P = 1/2(1+ \mathbf{n})$ (see probelm 1.11) with **n** a real unit vector, and g is the golden ratio (see previous problem).
a) prove that γ is antiunimodular, i.e. that $\gamma\bar{\gamma} = -1$ and $\gamma\mathbf{n}$ is unimodular.
b) Find γ^2 and use it to show that $\gamma^{n+1} = \gamma^n + \gamma^{n-1}$
c) Demnostrate that any integer power γ^n of γ can be expressed in the form $\gamma^n = a_n\gamma + a_{n-1}$ where a_n is the *n*th member of the Fibonacci sequence defined iteratively by $a_0=0$, $a_1=1$, $a_{n+1} = a_n + a_{n-1}$.
d) From the definition of γ , prove that $\gamma^n = g^nP + (-1/g)^n\bar{P}$ and use the limit of this expression for large n to show that lim (n → ∞) $a_{n+1}/a_n = g$

Sol 1.16 a) γ is antiunimodular if $\gamma\bar{\gamma} = \bar{\gamma}\gamma = -1$

$\gamma\bar{\gamma} = (gP - g^{-1}\bar{P})(g\bar{P} - g^{-1}P) = g^2\underbrace{P\bar{P}}_{\emptyset} - P^2 - \bar{P}^2 + g^{-2}\underbrace{\bar{P}P}_{\emptyset} = -(P+\bar{P}) = -1$

since $\bar{P}P = P\bar{P} = 0$ and $P^2 = P$, $\bar{P}^2 = \bar{P}$ and $P + \bar{P} = 1$

$(\gamma n)(\overline{\gamma n}) = (\gamma n)(\bar{\gamma} n) = (gP - g^{-1}\bar{P})n(g\bar{P} - g^{-1}P)n =$

since n is real

$= \left(g\frac{1}{2}(1+n)n - g^{-1}\frac{1}{2}(1-n)n\right)\left(g\frac{1}{2}(1-n)n - g^{-1}\frac{1}{2}(1+n)n\right) =$

$= \frac{1}{2}\left(g(n+1) - g^{-1}(n-1)\right)\frac{1}{2}\left(g(n-1) - g^{-1}(n+1)\right) =$

$= \frac{1}{4}\left(g^2\underbrace{(n^2-1)}_{\emptyset} - (n^2-2n+1) - (n^2+2n+1) + g^{-2}\underbrace{(n^2-1)}_{\emptyset}\right) = \frac{1}{4}(-4) = -1$

b) g is the golden ratio  $\frac{a}{b} = \frac{b-a}{a}$ $\frac{1}{g} = g - 1$ $g^2 - g - 1 = 0$

$\gamma^2 = (gP - g^{-1}\bar{P})(gP - g^{-1}\bar{P}) = g^2P^2 - \underbrace{P\bar{P}}_{\emptyset} - \underbrace{\bar{P}P}_{\emptyset} + g^{-2}\bar{P}^2 = g^2P^2 + g^{-2}\bar{P}^2$

$\left.\begin{array}{l} g^2 = 1+g \\ 1 = g^{-2} + g^{-1} \end{array}\right\} \Rightarrow \gamma^2 = \underbrace{(1+g)}_{g^2}\underbrace{P^2}_{= P} + \underbrace{(1-g^{-1})}_{g^{-2}}\underbrace{\bar{P}^2}_{= \bar{P}} = P + \bar{P} + (gP - g^{-1}\bar{P}) =$

$= 1 + \gamma$

$\gamma^3 = \gamma\gamma^2 = \gamma(1+\gamma) = \gamma + \gamma^2$; $\gamma^4 = \gamma(\gamma^3) = \gamma(\gamma + \gamma^2) = \gamma^2 + \gamma^3$

$\gamma^n = \gamma^{n-2} + \gamma^{n-1} \Rightarrow \gamma^{n+1} = \gamma(\gamma^{n-2} + \gamma^{n-1}) = \gamma^{n-1} + \gamma^n$ induction

c) $\gamma^3 = \gamma\gamma^2 = \gamma + (1+\gamma) = 2\gamma + 1 = a_3\gamma + a_2$

$\gamma^4 = (a_3\gamma + a_2)\gamma = a_3\gamma^2 + a_2\gamma = a_3(1+\gamma) + a_2\gamma = +\underbrace{(a_3 + a_2)}_{a_4}\gamma + a_3$

$\gamma^n = a_n\gamma + a_{n-1}$ assumed true for induction

$\gamma^{n+1} = \gamma(a_n\gamma + a_{n-1}) = a_n\gamma^2 + \gamma a_{n-1} = a_n(1+\gamma) + \gamma a_{n-1} = \gamma\underbrace{(a_n + a_{n-1})}_{a_{n+1} \text{ by definition}} + a_n$

d) from b) $\gamma^2 = g^2P^2 + g^{-2}\bar{P}^2$; $\gamma^3 = (gP - g^{-1}\bar{P})(g^2P^2 + g^{-2}\bar{P}^2) =$

$= g^3\underbrace{P^3}_{P} - g^{-3}\underbrace{\bar{P}^3}_{\bar{P}} + g^{-1}\underbrace{P\bar{P}^2}_{0} - g\underbrace{\bar{P}P^2}_{0} = g^3P + (-g^{-1})^3\bar{P}$

using induction $\gamma^n = g^nP + (-g^{-1})^n\bar{P}$

for $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ it is easier to use analysis: $\underbrace{\frac{a_{n+1}}{a_n}}_{= l} = \frac{a_n + a_{n-1}}{a_n}$

if limit exist and is l ⇒ $l = 1 + l^{-1}$ and by induction $l^2 - l - 1 = 0$

$1 < \frac{a_{n+1}}{a_n} < 2$ ⇒ use only positive solution of ⇒ $l = g$